ARTICLE NO. 80-4117

Discrete-Continuous Variable Structural Synthesis Using Dual Methods

Lucien A. Schmit* and Claude Fleury†
University of California, Los Angeles, Calif.

Approximation concepts and dual methods are extended to solve structural synthesis problems involving a mix of discrete and continuous sizing type of design variables. Pure discrete and pure continuous variable problems can be handled as special cases. The basic mathematical programming statement of the structural synthesis problem is converted into a sequence of explicit approximate primal problems of separable form. These problems are solved by constructing continuous explicit dual functions, which are maximized subject to simple nonnegativity constraints on the dual variables. A newly devised gradient projection type of algorithm called DUAL 1, which includes special features for handling dual function gradient discontinuities that arise from the discrete primal variables, is used to find the solution of each dual problem. Computational implementation is accomplished by incorporating the DUAL 1 algorithm into the ACCESS 3 program as a new optimizer option. The power of the method set forth is demonstrated by presenting numerical results for several example problems, including a pure discrete variable treatment of a metallic swept wing and a mixed discrete-continuous variable solution for a thin delta wing with fiber composite skins.

Introduction

THERE are many occasions in structural optimization where the design variables describing the member sizes must be selected from a list of discrete values. For example, conventional metal alloy sheets are commercially available in standard gage sizes and cross-sectional areas for truss members may, in practice, have to be chosen from a list of commercially available member sizes. Furthermore, the growing use of fiber composite materials in aerospace structures also underscores the importance of being able to treat structural synthesis problems where some or even all of the design variables are discrete.

In the structural optimization literature relatively little attention has been given to dealing with discrete variables. Those efforts that have been reported (see Ref. 1 for a review of this literature) generally attack the discrete variable design optimization problem by employing discrete or integer variable algorithms to treat the problem directly in the primal variable space (e.g., see Refs. 2-4). In this paper it will be shown that the combined use of approximation concepts and dual methods, set forth in Ref. 5 for continuous sizing type design variables, can be extended to structural synthesis problems involving a mix of discrete and continuous sizing type design variables.

The basic approach followed here (and in Ref. 5) is outlined schematically in Fig. 1. Approximation concepts convert the basic mathematical programming statement of the structural synthesis problem into a sequence of explicit problems of separable form. An explicit dual function corresponding to each approximate primal problem is constructed and its maximum is sought, subject to nonnegativity constraints on the dual variables, using a specially devised first-order gradient projection type of algorithm called DUAL 1. The primary computational effort is concentrated on finding the maximum of the dual function, since the values of the primary variables are easily determined once the dual variable values corresponding to the dual function maximum point are known. The mixed-case formulation and the implementing algorithm DUAL 1 presented here can also handle the two limiting special cases, namely, the pure discrete and the pure continuous variable cases.

Formulation

The class of structural synthesis problems addressed in this paper involves sizing type design variables for thin-walled structures modeled by bar, shear panel and membrane elements. It is now well known that this significant class of problems can be approximated accurately by a primal mathematical programming problem having a special algebraic structure, namely, separable objective function and linear inequality constraints. These approximate primal problems are generated through the coordinated use of approximation concepts which: 1) reduce the number of primal design variables via linking; 2) temporarily reduce the number of inequality constraints by employing constraint deletion techniques, so that only critical and potentially critical constraints are retained at any given stage in the iterative design process; and 3) reduce the number of complete finite-element structural analyses by constructing linear approximations for retained constraints in terms of linked reciprocal design variables.

At each stage of the overall iterative design process outlined in Fig. 1, the approximate primal problem for the general mixed-variable case, where some of the sizing design variables are discrete and others are continuous, has the following form:

Find $\boldsymbol{\alpha}$ such that

$$W(\boldsymbol{\alpha}) = \sum_{b=1}^{B} \frac{w_b}{\alpha_b} \rightarrow \text{Min} \tag{1}$$

subject to

$$\tilde{h}_q(\boldsymbol{\alpha}) = \bar{u}_q - u_q(\boldsymbol{\alpha}) \geq 0; \quad q \in Q_R \tag{2}$$

where

$$u_q(\boldsymbol{\alpha}) = \sum_{b=1}^{B} C_{bq}\alpha_b \tag{3}$$

Received July 30, 1979; revision received April 7, 1980.

Index categories: Structural Design; Analytical and Numerical Methods; Structural Statics.

*Professor of Engineering and Applied Science, Mechanics and Structures Dept. Associate Fellow AIAA.

†Postdoctoral Research Fellow, Mechanics and Structures Dept. and Aspirant FNRS, Belgium.

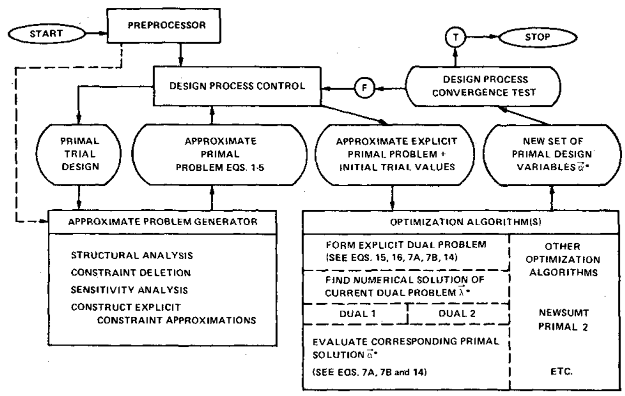

Fig. 1 Basic organization of ACCESS 3.

and with side constraints

$$\alpha_b^{(L)} \leq \alpha_b \leq \alpha_b^{(U)} \quad \text{for } b \in B_C \tag{4}$$

and

$$\alpha_b \in A_b^{(D)} = \{\alpha_b^{(k)};\ \ k=1,2,...n_b\} \ \text{ for } b\in B_D \tag{5}$$

It is to be understood that B_C denotes the set of indices b for continuous design variables while B_D denotes the set of indices b for discrete design variables. The w_b in Eq. (1) are positive fixed constants corresponding to the weight of the finite elements in the bth linking group when $\alpha_b = 1$, and Eqs. (2) and (3) represent the current linearized approximations of the retained behavior constraints, in which the C_{bq} are constants. The $\alpha_b^{(L)}$ and $\alpha_b^{(U)}$ in Eq. (4), respectively, denote the lower and upper limits on those independent reciprocal design variables that are continuous (i.e., $b\in B_C$) and Eq. (5) may be viewed as representing the side constraints on those independent reciprocal design variables that are discrete (i.e., $b\in B_D$). It is to be understood that the symbol $A_b^{(D)}$ represents the set of available discrete values for the design variable α_b, listed in ascending order. For convenience the index p denoting the stage in the overall iterative design process has been dropped from Eqs. (1-5). However, it should be kept in mind that in general Eqs. (1-5) represent only the approximate primal problem for the pth stage of the overall iterative design process outlined in Fig. 1.

It was shown in Sec. 2.4.2 of Ref. 6 that separability is the key to being able to develop an explicit dual function, corresponding to the approximate primal problem at each stage. For the pure continuous variable case dealt with in Ref. 5, it was shown that the primal variables are given implicitly by

$$\underset{\alpha_b^{(L)} \leq \alpha_b \leq \alpha_b^{(U)}}{\text{Min}} \left\{ \frac{w_b}{\alpha_b} + \alpha_b \sum_{q\in Q_R} \lambda_q C_{bq} \right\};\ b\in B_C \tag{6}$$

and explicitly by

$$\alpha_b = \begin{cases} \alpha_b^{(L)} & \text{if } [\alpha_b^{(L)}]^2 \geq \tilde{\alpha}_b^2 \\ \tilde{\alpha}_b & \text{if } [\alpha_b^{(L)}]^2 < \tilde{\alpha}_b^2 < [\alpha_b^{(U)}]^2 \\ \alpha_b^{(U)} & \text{if } \quad \tilde{\alpha}_b^2 \geq [\alpha_b^{(U)}]^2 \end{cases} \tag{7a}$$

where

$$\tilde{\alpha}_b^2 = w_b \Big/ \sum_{q\in Q_R} \lambda_q C_{bq} \tag{7b}$$

In an analogous manner, for discrete α_b it is assumed that

$$\underset{\alpha_b \in A_b^{(D)}}{\text{Min}} \left\{ \frac{w_b}{\alpha_b} + \alpha_b \sum_{q\in Q_R} \lambda_q C_{bq} \right\};\ b\in B_D \tag{8}$$

relates the continuous dual variables to the discrete primal variables $\alpha_b;\ b\in B_D$.‡ Continuing the analogy and drawing on the expressions for the dual function and its first derivatives, given respectively by Eqs. (5) and (9) of Ref. 5, leads to the following expressions for the mixed-case dual function

$$\ell(\lambda) = \sum_{b\in B_C} \left(\frac{w_b}{\alpha_b}\right) + \sum_{b\in B_D} \left(\frac{w_b}{\alpha_b}\right) + \sum_{q\in Q_R} \lambda_q [u_q(\alpha) - \bar{u}_q] \tag{9}$$

and its first derivatives

$$\frac{\partial \ell}{\partial \lambda_q}(\lambda) = u_q(\alpha) - \bar{u}_q = \sum_{b\in B_C} C_{bq}\alpha_b + \sum_{b\in B_D} C_{bq}\alpha_b - \bar{u}_q \tag{10}$$

It is apparent from Eq. (10) that discrete values for the $\alpha_b;\ b\in B_D$ will cause discontinuities in the first derivatives of the dual function. When the solution of Eq. (8) shifts from one discrete value $\alpha_b^{(k)}$ to the next $\alpha_b^{(k+1)}$, the following identity maintains continuity of the dual function

$$\frac{w_b}{\alpha_b^{(k)}} + \alpha_b^{(k)} \sum_{q\in Q_R} \lambda_q C_{bq} = \frac{w_b}{\alpha_b^{(k+1)}} + \alpha_b^{(k+1)} \sum_{q\in Q_R} \lambda_q C_{bq} \tag{11}$$

Equation (11) can be reduced to the following form

$$\sum_{q\in Q_R} \lambda_q C_{bq} = \frac{w_b}{\alpha_b^{(k)}\alpha_b^{(k+1)}} \tag{12}$$

which defines hyperplanes in the dual space where the dual function $\ell(\lambda)$ exhibits first-order discontinuities. The hyperplane defined by Eq. (12) subdivides the dual space into a region where $\alpha_b = \alpha_b^{(k)}$ and another region where $\alpha_b = \alpha_b^{(k+1)}$. Similarly the hyperplane defined by

$$\sum_{q\in Q_R} \lambda_q C_{bq} = \frac{w_b}{\alpha_b^{(k-1)}\alpha_b^{(k)}} \tag{13}$$

is associated with a shift in the solution of Eq. (8) from $\alpha_b^{(k-1)}$ to $\alpha_b^{(k)}$ and it subdivides the dual space into a region where $\alpha_b = \alpha_b^{(k-1)}$ and another region where $\alpha_b = \alpha_b^{(k)}$. It follows from the foregoing interpretation of Eqs. (12) and (13), that the discrete α_b are explicitly related to the continuous dual variables as follows:

$\alpha_b = \alpha_b^{(k)};\ b\in B_D$ if the λ_q values are such that

$$\frac{w_b}{\alpha_b^{(k-1)}\alpha_b^{(k)}} < \sum_{q\in Q_R} \lambda_q C_{bq} < \frac{w_b}{\alpha_b^{(k)}\alpha_b^{(k+1)}} \tag{14}$$

In summary, the explicit dual problem corresponding to the general mixed-variable approximate primal problem posed by Eqs. (1-5) is taken to have the form:

Find λ such that

$$\ell(\lambda) = \left(\sum_{b\in B_C} \frac{w_b}{\alpha_b}\right) + \left(\sum_{b\in B_D} \frac{w_b}{\alpha_b}\right) + \sum_{q\in Q_R} \lambda_q \left[\left(\sum_{b\in B_C} C_{bq}\alpha_b\right) + \left(\sum_{b\in B_D} C_{bq}\alpha_b\right) - \bar{u}_q\right] \to \text{Max} \tag{15}$$

‡This pragmatic extension of the dual formulation is not rigorous from a strict mathematical point of view because the approximate primal problem is no longer convex when discrete variables are introduced.

subject to simple nonnegativity constraints

$$\lambda_q \geq 0; \quad q \in Q_R \tag{16}$$

where the continuous primal variables (α_b; $b \in B_C$) are given in terms of the dual variables by Eqs. (7) and the discrete primal variables (α_b; $b \in B_D$) are related to the dual variables by Eq. (14).

Characteristics of Dual Function

The explicit dual function for the mixed discrete-continuous variable problem, defined by Eqs. (15), (7), and (14), has the following interesting and computationally important properties:

1) It is a concave function and the search region in dual space is a convex set defined by Eq. (16).

2) It is a continuous function and it has continuous first derivatives with respect to λ_q over the region defined by Eq. (16) *except* for points located in hyperplanes defined by Eq. (12)—these first-order discontinuities are associated with shifts in the discrete variable solution of the one-dimensional minimization problem represented by Eq. (8).

3) The first derivatives of $\ell(\lambda)$ are easily available because they are given by the primal constraints [see Eq. (10)] and on the first-order discontinuity planes two distinct values of the first derivative arise, because at such a point there is a shift in the discrete value of a particular primal variable, say α_b, from $\alpha_b^{(k)}$ to $\alpha_b^{(k+1)}$ which gives [see Eq. (10)]:

$$\frac{\partial \ell^{(k)}}{\partial \lambda_q}(\lambda) = \sum_{j \in B_C} C_{jq}\alpha_j + \sum_{\substack{j \in B_D \\ j \neq b}} C_{jq}\alpha_j + C_{bq}\alpha_b^{(k)} - \bar{u}_q \tag{17}$$

and

$$\frac{\partial \ell^{(k+1)}}{\partial \lambda_q}(\lambda) = \sum_{j \in B_C} C_{jq}\alpha_j + \sum_{\substack{j \in B_D \\ j \neq b}} C_{jq}\alpha_j + C_{bq}\alpha_b^{(k+1)} - \bar{u}_q \tag{18}$$

4) Discontinuities of the second derivatives $(\partial^2 \ell / \partial\lambda_q \partial\lambda_k)(\lambda)$ exist on hyperplanes in the dual space defined by [see Eqs. (20) and (21) of Ref. 5]

$$\sum_{q \in Q_R} \lambda_q C_{bq} = \frac{w_b}{[\alpha_b^{(L)}]^2}; \quad b \in B_C \tag{19a}$$

and

$$\sum_{q \in Q_R} \lambda_q C_{bq} = \frac{w_b}{[\alpha_b^{(U)}]^2}; \quad b \in B_C \tag{19b}$$

for continuous variables α_b; $b \in B_C$—these second-order discontinuities locate points in the dual space where there is a change in status of the bth continuous primal variable from "free" to "bound." A continuous primal variable is said to be "free" if it has not taken on its upper or lower bound value ($\alpha_b^{(U)}$ or $\alpha_b^{(L)}$) [see Eq. (7)].

In the pure discrete case the explicit dual function is piecewise linear, that is, the contours are sections of intersecting hyperplanes. The dual space is partitioned into several domains each of which corresponds to a distinct combination of available discrete values of the primal variables. A two-dimensional example which illustrates this observation graphically will be found in Sec. 4.3 of Ref. 6.

The main difficulty associated with the explicit dual formulation for the mixed discrete-continuous variable problem is linked to the existence of hyperplanes in the dual space [see Eq. (12)] where the gradient of the dual function $\nabla\ell(\lambda)$ is not uniquely defined, because of the previously described first-order discontinuities [see Eqs. (17) and (18)]. These first-order discontinuity hyperplanes complicate the task of finding the maximum of the dual function. Fortunately, it turns out that at points in the dual space where the gradient $\nabla\ell(\lambda)$ is multivalued, the projection of each distinct gradient into the subspace defined by the set of pertinent discontinuity hyperplanes yields a single move direction $\boldsymbol{Z}$. Furthermore, it can be shown that the directional derivative $(d\ell/dz)$ of the dual function along the direction $\boldsymbol{Z}$ is unique and positive.

An intuitive grasp of the basic scheme used to cope with the first-order discontinuities can be gained by considering the simple case of a single-discontinuity plane, which is depicted schematically in Fig. 2. Let the equation of the first-order discontinuity plane (line a-a in Fig. 2) be represented by [see Eq. (12) with $k=1$]

$$f_b(\lambda) = \lambda^T \boldsymbol{C}_b - (w_b / \alpha_b^{(1)}\alpha_b^{(2)}) = 0 \tag{20}$$

then the normal to the discontinuity plane is

$$\nabla f_b = \boldsymbol{C}_b \tag{21}$$

Let $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ denote the two distinct values of the gradient at point t in Fig. 2. Referring to Eqs. (17) and (18) and setting $k=1$ it follows that

$$\boldsymbol{g}_1 = \boldsymbol{h} + \alpha_b^{(1)}\boldsymbol{C}_b \tag{22}$$

and

$$\boldsymbol{g}_2 = \boldsymbol{h} + \alpha_b^{(2)}\boldsymbol{C}_b \tag{23}$$

where it is understood that the components of the vector $\boldsymbol{h}$ are given by

$$h_q = \sum_{j \in B_C} C_{jq}\alpha_j + \sum_{\substack{j \in B_D \\ j \neq b}} C_{jq}\alpha_j - \bar{u}_q \tag{24}$$

The projections of $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ into the discontinuity plane are given by

$$\boldsymbol{Z} = \boldsymbol{g}_1 - (\boldsymbol{C}_b^T\boldsymbol{g}_1 / \boldsymbol{C}_b^T\boldsymbol{C}_b)\boldsymbol{C}_b \tag{25}$$

and

$$\boldsymbol{Z} = \boldsymbol{g}_2 - (\boldsymbol{C}_b^T\boldsymbol{g}_2 / \boldsymbol{C}_b^T\boldsymbol{C}_b)\boldsymbol{C}_b \tag{26}$$

To confirm that the move direction $\boldsymbol{Z}$ given by Eqs. (25) and (26) is unique, simply substitute Eq. (22) in Eq. (25) or Eq.

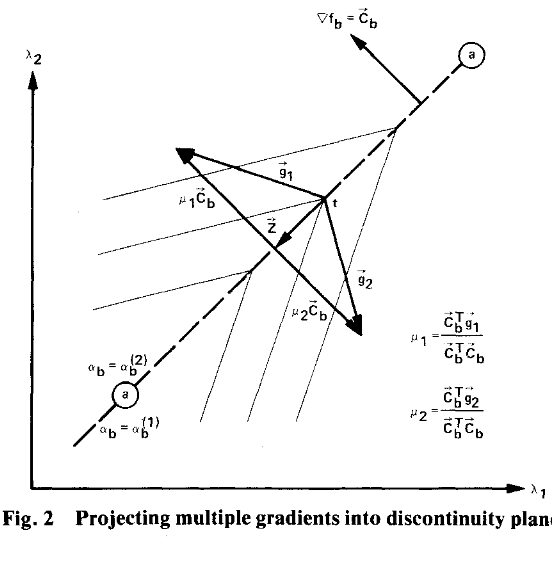

Fig. 2 Projecting multiple gradients into discontinuity plane.

(23) into Eq. (26) to find that

$$Z = h - \frac{C_b^T h}{C_b^T C_b} C_b = \left[I - \frac{C_b C_b^T}{C_b^T C_b} \right] h \tag{27}$$

in either case. Furthermore, using Eqs. (22), (23), and (27) it is easily shown that the directional derivative along Z is unique and positive, provided $Z \neq 0$, since

$$\frac{d\ell}{dz} = Z^T g_1 = Z^T g_2 = h^T h - \frac{(C_b^T h)^2}{C_b^T C_b} = Z^T Z > 0 \tag{28}$$

The foregoing development can be generalized to the case where the current point in the dual space λ_t resides in a subspace defined by F first-order discontinuity hyperplanes [see Eq. (12)]. For convenience assume that the primal variables are numbered so that the first F are discrete variables associated with a distontinuity hyperplane pertinent to the current point in dual space λ_t. At such a point in dual space there are 2^F different gradients corresponding to the 2^F possible combinations of the values of $\alpha_b^{(k)}$ and $\alpha_b^{(k+1)}$ for $b = 1,2,\ldots F$ and they can be represented as follows

$$g^{(\ell)} = h + \sum_{b=1}^{F} \alpha_b^{(\ell)} C_b; \quad \ell = 1,2,\ldots 2^F \tag{29}$$

where the components of h are given by

$$h_q = \sum_{j \in B_C} C_{jq} \alpha_j + \sum_{\substack{j \in B_D \\ j > F}} C_{jq} \alpha_j - \bar{u}_q \tag{30}$$

and the vectors C_b denote the gradients to the pertinent set of F discontinuity hyperplanes. It can be shown that projection of the 2^F distinct gradients $g^{(\ell)}$ represented in Eq. (29) into the subspace defined by the F discontinuity hyperplanes yields a unique direction of travel (see Sec. 4.4 of Ref. 6)

$$Z = [P] h \tag{31}$$

where the projection matrix is given by

$$[P] = [I - N(N^T N)^{-1} N^T] \tag{32}$$

in which I is a $Q_R \times Q_R$ identity matrix and N denotes a $Q_R \times F$ matrix with columns corresponding to the vectors $C_1, C_2, \ldots C_b \ldots C_F$ appearing in Eq. (29), that is

$$[N] = [C_1, C_2, \ldots C_b \ldots C_F] \tag{33}$$

Furthermore, it can be shown that the directional derivative along the move direction Z given by Eq. (31) is unique and positive (see Sec. 4.4 of Ref. 6), that is

$$\frac{d\ell}{dz} = Z^T g^{(\ell)} = Z^T Z > 0 \text{ if } Z \neq 0 \tag{34}$$

In the DUAL 1 algorithm described in the next section, direction vectors are generated using the projection matrix $[P]$, whenever the current point λ resides in one or more first-order discontinuity hyperplanes. However, for computational efficiency the $[P]$ matrix is actually generated by employing update formulas rather than by using Eq. (32).

Maximization Algorithm—DUAL 1

In this section a first-order gradient projection type of algorithm for finding the maximum of the explicit dual function $\ell(\lambda)$, defined by Eqs. (15), (7), and (14) subject to the nonnegativity constraints of Eq. (16), is described. The existence of hyperplanes in the dual space where the dual function exhibits first-order discontinuities [see Eq. (12) or (13)] requires the use of a specially devised first-order algorithm called DUAL 1, which is akin to the well-known gradient projection method. For each stage p in the overall iterative design process, the DUAL 1 algorithm seeks λ such that $\ell(\lambda) \to$ Max subject to $\lambda_q \geq 0$; $q \in Q_R^{(p)}$. The dual variable vector is modified iteratively as follows

$$\lambda_{t+1} = \lambda_t + d_t S_t \tag{35}$$

and the maximization algorithm consists of a sequence of one-dimensional maximizations (ODMs) executed along ascent directions S_t obtained by projecting the dual function gradient into an appropriate subspace.

The DUAL 1 algorithm for the general mixed-variable case (discrete and continuous variables) is described using the schematic block diagram shown in Fig. 3. At each step the direction S_t is taken as the gradient $\nabla \ell(\lambda_t)$ or a projection of the gradient into an appropriate subspace. The scheme for generating the next search direction depends upon the nature of the previous ODMs termination point.

Initially [block 1, Fig. 3] or when the dual point (λ_t) does not reside in any of the discontinuity planes, the move direction S_t is taken as the gradient at λ_t modified so as to avoid violation of the nonnegativity constraints $\lambda_q \geq 0$; $q \in Q_R$, that is (block 2)

$$S_{qt} = 0 \text{ if } \lambda_{qt} = 0 \text{ and } \frac{\partial \ell}{\partial \lambda_q}(\lambda) = u_q(\alpha_t) - \bar{u}_q \leq 0 \tag{36}$$

and

$$S_{qt} = \frac{\partial \ell}{\partial \lambda_q}(\lambda_t) = u_q(\alpha_t) - \bar{u}_q \text{ otherwise} \tag{37}$$

The foregoing procedure for generating the move vector is equivalent to projecting the gradient vector into the subspace represented by the set of base planes $\lambda_q = 0$; $q \in N$ where

$$N = \left\{ q \mid \lambda_{qt} = 0; \ \frac{\partial \ell}{\partial \lambda_q}(\lambda_t) \leq 0; \ q \in Q_R \right\} \tag{38}$$

Typically, at this point, the convergence test $|S_t| < \epsilon$ (block 3) will not be satisfied, therefore go to block 4 and determine whether or not the conjugate direction modification is appropriate.

Whenever it makes sense, successive move directions are conjugated to each other using the well-known Fletcher-Reeves formula (see Ref. 7, p. 87)

$$S_t \leftarrow S_t + \beta_t S_{t-1} \tag{39}$$

where

$$\beta_t = S_t^T S_t / S_{t-1}^T S_{t-1} \tag{40}$$

The second and subsequent move directions within a subspace are generated using Eqs. (39) and (40) (block 5). In the DUAL 1 algorithm several ODMs can take place without a change in subspace, provided they do not terminate on either a new first-order discontinuity plane or a new base plane. In any event the conjugacy modification is reinitialized ($\beta_t = 0$) if the number of ODMs executed within a single subspace becomes equal to the dimensionality of the subspace. The dimensionality of a subspace is equal to Q_R, less the number of zero dual variables N, less the number of first-order discontinuity planes encountered so far.

With the move direction S_t established in block 4 or 5, go to block 6 and solve the ODM problem. The scheme employed to solve the ODM problem in DUAL 1 will be described subsequently. After solving the current ODM (block 6), there are six possible paths leading to the calculation of a new move

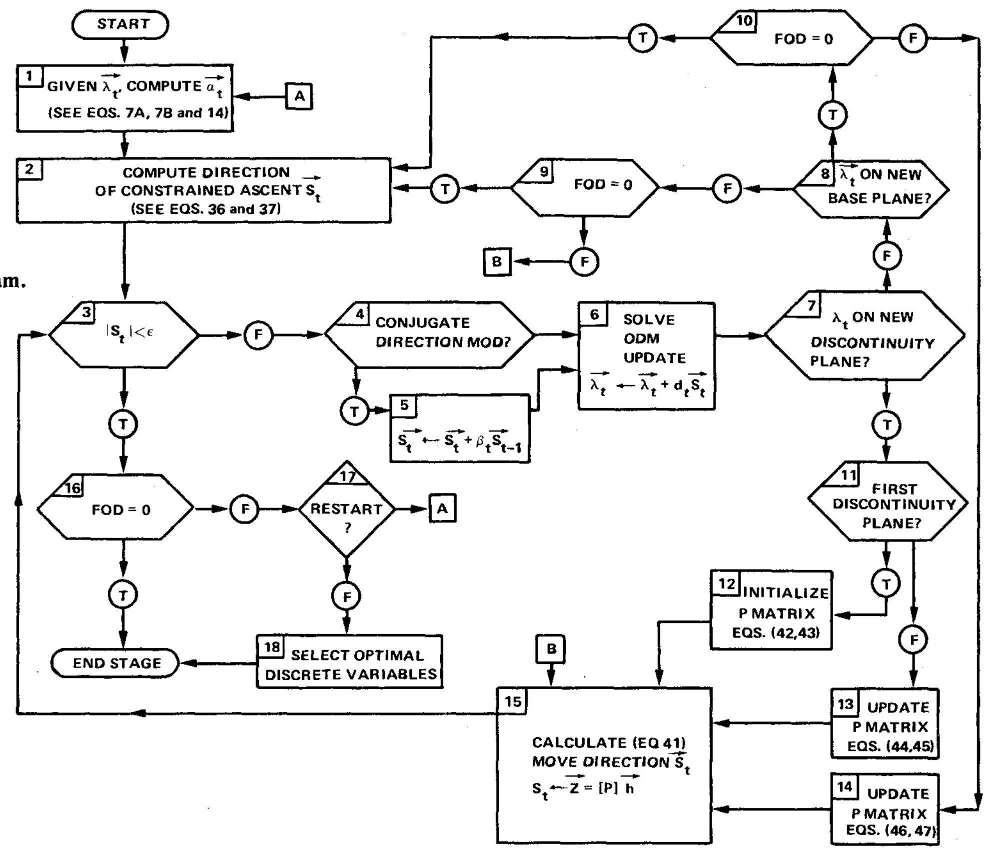

Fig. 3 DUAL 1 algorithm block diagram.

direction S_t in either block 15 where

$$S_t \leftarrow Z = [P]h \tag{41}$$

or block 2 where the move direction S_t is generated using Eqs. (36) and (37). These paths can be traced in Fig. 3 and they are described in detail in Sec. 4.5.1 of Ref. 6. Due to space limitations, only the most significant features are considered in this paper. In block 12, the projection matrix is initialized according to the following procedure: let C_b denote the gradient to the first discontinuity plane encountered. Construct a trial projection matrix

$$[P] = [I] - C_b C_b^T / C_b^T C_b \tag{42}$$

and project either value of the gradient at λ_t into the subspace defined by the first discontinuity plane using Eq. (31), that is

$$Z_t = [P]h \tag{43}$$

If $Z_{qt} < 0$ for $\lambda_{qt} = 0$ set the corresponding elements of the vector C_b to zero ($C_{bq} = 0$) and recalculate $[P]$ and Z_t. When $Z_{qt} \geq 0$ for all $\lambda_{qt} = 0$ the initial projection matrix has been obtained. The end result of this iteration is to generate a $[P]$ matrix that projects any vector into the subspace defined by the first discontinuity plane and the appropriate current set of $\lambda_q = 0$ base planes.

Whenever a new discontinuity plane is encountered, the projection matrix is updated in block 13 by letting

$$Y \leftarrow [P] C_b \tag{44}$$

where C_b is understood to be the gradient to the new discontinuity plane, and then modifying the $[P]$ matrix as follows:

$$[P] \leftarrow [P] - YY^T / Y^T Y \tag{45}$$

Finally, in block 14, the $[P]$ matrix is updated as follows:

$$Y \leftarrow [P] e_q \tag{46}$$

where e_q is a unit vector normal to the newly encountered base plane and

$$[P] \leftarrow [P] - YY^T / Y^T Y \tag{47}$$

After solving the ODM and updating λ_t (block 6, Fig. 3) the result is a new move direction vector S_t. If the new move direction vector has an absolute value equal to or greater than ϵ (block 3→F) the search for the maximum of the dual function in the current subspace continues (i.e., go to block 4). On the other hand, if $|S_t| < \epsilon$ or if the subspace defined by the set of base planes $\lambda_q = 0$; $q \epsilon N$ and the set of F first-order discontinuity planes has collapsed to a single point (i.e., $Q_R = N + F$), go to block 16. If no first-order discontinuity planes have been encountered (i.e., $FOD = 0 \rightarrow T$) then the maximum of the dual function, subject to the nonnegativity constraints $\lambda_q \geq 0$; $q \epsilon Q_R$, has been obtained, the stage is complete, and the values of the primal variables are stored.

On the other hand, if one or more first-order discontinuity planes have been encountered (block 16→F), go to block 17 and test to see whether or not the algorithm must be restarted. Provision for restart must be included because the foregoing updating scheme does not provide for the release of discontinuity or base plane equality constraints. Therefore, the maximization process can terminate at a dual point that is not necessarily the optimum. The DUAL 1 algorithm copes with this difficulty by restarting the maximization procedure, dropping all of the previously accumulated base and discontinuity plane constraints. If the true maximum has not yet been attained, then the algorithm will sequentially accumulate a new set of discontinuity and base planes and terminate at a different point with a higher dual function value. On the other hand, if the previous maximum of the dual function was really the true dual optimum, then the new updating sequence will generate the same projection matrix and the dual function maximum will be located in the same subspace as before. The restart test is as follows: compare the current value of the dual function $\ell(\lambda_t)$ with its value when the restart block 17 was previously entered. If the difference is small, go to block 18; otherwise, return to block 1 via point A in Fig. 3 and restart releasing all of the previously accumulated equality constraints. It should be noted that unless the stage ends without encountering any discontinuity planes (block 16→T) there will be at least one restart. Computational experience indicates that less than five restarts are usually required to locate the dual maximum.

For mixed discrete-continuous variable problems a stage usually ends by exiting block 17→F and entering block 18 with a dual point λ_t that resides on one or more first-order discontinuity planes [see Eq. (12)]. For each of these F discontinuity planes the corresponding primal variable α_b has two candidate values denoted $\alpha_b^{(k)}$ and $\alpha_b^{(k+1)}$. The upper bound solution is obtained by selecting the smaller discrete value for each such reciprocal variable. If the upper bound design is feasible (feasible with respect to the approximate constraints for the pth stage), the lowest weight feasible design is selected from the set of 2^F possibilities existing. On the other hand, if the upper bound solution is not feasible, then a feasible design or by default the design which is most nearly feasible is selected from the set of 2^F possibilities. This is done by finding the design for which the most seriously violated constraint exhibits the smallest infeasibility. The foregoing discrete search through 2^F possible designs is organized in such a way that when passing from one primal candidate design to the next, only one design variable changes. As a consequence, the new weight and the associated constraint values can be computed very efficiently in view of the separability of the weight and constraint functions [see Eqs. (1) and (3)].

It is important to recall that the number F of first-order discontinuity planes cannot be larger than the number $(Q_R - N)$ of nonzero dual variables. In other words the number of ambiguous design variables, for which one out of two candidate values must be selected, never exceeds the number of strictly critical behavior constraints, which is precisely the operational dimensionality of the dual space. This number is usually small for practical problems and therefore dual methods retain their attractiveness when discrete variables are introduced.

In the DUAL 1 algorithm, once a move direction is established (see blocks 2, 15, 4, and 5 of Fig. 3), it is necessary to find the maximum of the explicit dual function $\ell(\lambda)$ along the direction S_t [see Eq. (35)]. It follows that $\ell(\lambda)$ becomes a function of the scalar move distance d once S_t is established. The ODM of the concave function $\ell(d)$ is equivalent to seeking the vanishing point of the directional derivative

$$\ell'(d) = S_t^T \nabla \ell(d) = \sum_{b=1}^{B} \alpha_b(d) \Big(\sum_{q \in Q_R} S_{qt} C_{bq} \Big) - \sum_{q \in Q_R} S_{qt} \bar{u}_q \quad (48)$$

The primal variables α_b are known explicit functions of d along the direction S_t in dual space [by using Eqs. (7) for continuous variables $b \in B_C$ and Eq. (14) for discrete variables $b \in B_D$].

The ODM problem consists of seeking the vanishing point of $\ell'(d)$ over the interval $0 < d < d_{max}$. The maximum allowable step length d_{max} is selected so that none of the dual variables can become negative, that is

$$d_{max} = \underset{S_{qt} < 0}{\mathrm{Min}} \left| \frac{\lambda_{qt}}{S_{qt}} \right| \quad (49)$$

The key idea of the ODM procedure used in DUAL 1 is that the intercept distances to the first- and second-order discontinuity planes [see Eqs. (12) and (19)] are evaluated and used to help locate the one-dimensional maximum. The procedure will be described qualitatively using Fig. 4 to help clarify the basic approach followed. Attention will be focused on the mixed discrete-continuous variable case (Fig. 4C). A more detailed presentation of the ODM solution scheme can be found in Sec. 4.4.4 of Ref. 6. A representative plot of the dual function vs d is shown in Fig. 4C (for the mixed discrete-continuous variable case). With d_{max} known from Eq. (49) the intercept distances d_j to both first- [Eq. (12)] and second-order [Eqs. (19)] discontinuity planes that intersect S_t between 0 and d_{max} (i.e., $0 < d_j < d_{max}$) are evaluated and stored in ascending order. The slope(s) of the dual function along S_t is evaluated at each intercept location until a sign change signals the location (see Fig. 4, C1) or trapping (see Fig. 4, C2) of the maximum. In the first case (Fig. 4, C1) the maximum has been located at d_j since $\ell'(d_j^-) > 0$ and $\ell'(d_j^+) < 0$. In the second case (Fig. 4, C2) the maximum has been trapped in the interval $d_j < d < d_{j+1}$ since $\ell'(d_j) > 0$ and $\ell'(d_{j+1}^-) < 0$. The Newton method is then used to locate the point d^* where $\ell'(d^*) = 0$ and this distance is taken as the solution of the ODM (i.e., $d_t \leftarrow d^*$).

It is important to notice that with the foregoing ODM procedure, the discrete primal variables do not need to be explicitly computed from Eq. (14), where a lot of tests have to be completed before finding the correct discrete values. They are directly deduced from "status" vectors that are computed, stored, and reordered when the intercept distances $(0 < d_j < d_{max})$ are evaluated and put in ascending order. As a result, it is only in block 1 of Fig. 3 that the discrete primal variables are evaluated using Eq. (14), i.e., in order to start or restart the maximization procedure. Subsequently, the primal variables are always determined in the ODM part of the DUAL 1 algorithm using the intercept values and status vectors.

Numerical Examples

The basic ideas set forth in this paper, extending dual methods to mixed discrete-continuous variable structural synthesis problems, have been implemented in a computer program called ACCESS§ 3. This research-type computer program was written by modifying the ACCESS 2 program,[8,9] essentially adding in the dual formulation as well as the DUAL 1 and DUAL 2 maximization algorithms. Herein, attention is focused on application of the DUAL 1 algorithm which is specially devised to cope with the first-order discontinuities which arise when discrete design variables are included. The DUAL 1 algorithm can handle pure discrete and pure continuous variable problems, since they are special cases of the general mixed discrete-continuous design variable problem. It should be noted, however, that the DUAL 2 algorithm presented in Ref. 5, which is applicable only to pure continuous variable problems, is more efficient. The scope of the ACCESS 3 program is summarized in Ref. 5 and more detailed descriptions are offered in Refs. 6 and 10. It is worth noting that the ACCESS 3 program contains all ACCESS 2[8,9] capabilities as a subset and the data preparation formats are fully compatible.[10]

Numerical results obtained with ACCESS 3 for various structural optimization problems involving discrete design variables are given in Ref. 6. Initially several truss problems are employed as test cases to validate the DUAL 1 optimizer. Among them the classical 10 bar truss cases deserve special mention here. Three cases involving discrete variables were created from the well-studied continuous variable solution in the following way. The available discrete values are selected as the natural integer sequence in which the optimal design variable values obtained in the pure continuous case have been inserted, i.e., the set of discrete values is {0.1, 0.551, 1.0,..., 7.0, 7.457, 8.0,..., 15.0, 15.22, 16.0,..., 21.0, 21.04,

§Approximation concepts code for efficient structural synthesis.

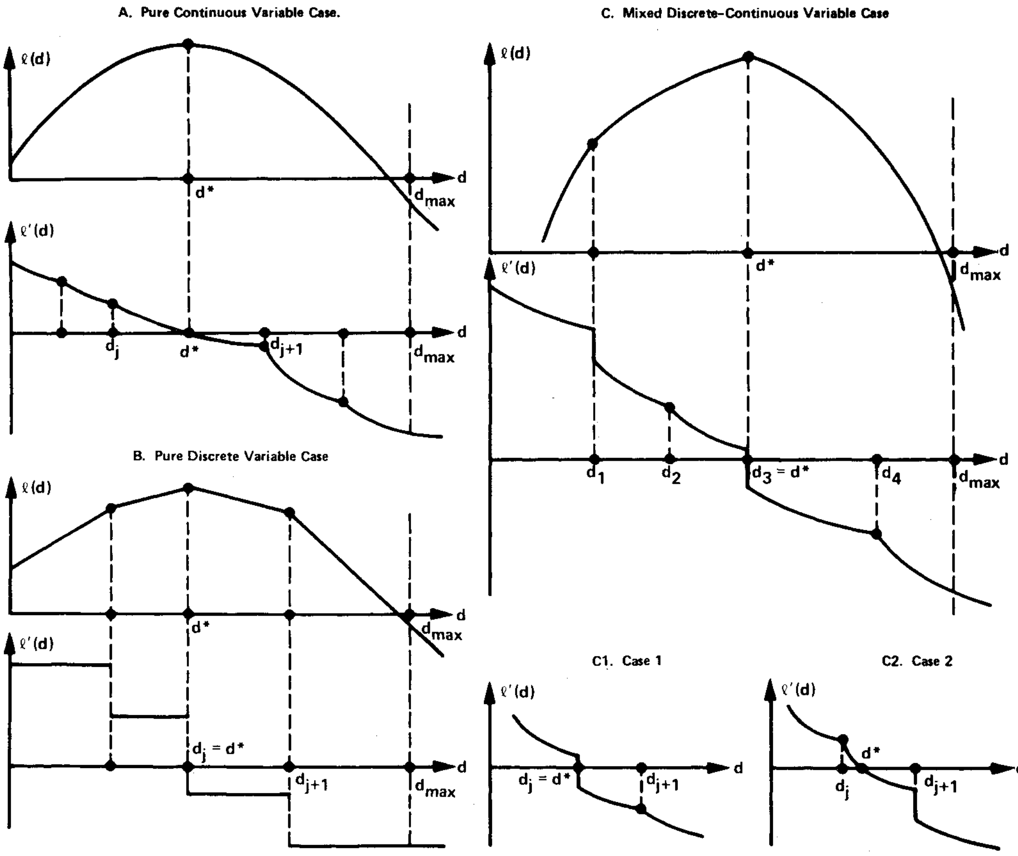

Fig. 4 One-dimensional maximization scheme.

21.53, 22.0, 23.0, 23.20, 24.0,..., 30.0, 30.52, 31.0,..., 40.0}. A pure discrete and two mixed continuous-discrete cases were treated. In all cases the DUAL 1 optimizer was capable of retrieving the expected optimal solution within 13-15 iterations. Subsequently, the DUAL 1 option in ACCESS 3 is applied to an idealized swept wing problem with pure discrete variables based on available standard gage sizes. Finally, the DUAL 1 option is applied to an idealized delta wing problem involving laminated fiber composite skins and a mix of discrete and continuous variables is employed. These results for discrete and mixed discrete-continuous variable structural synthesis problems show that although the extension of dual methods to this class of problems is lacking in mathematical rigor, the DUAL 1 algorithm represents a powerful practical design tool. The swept wing and delta wing examples will be presented here in brief summary form; more detailed data can be found in Ref. 6. All computations were carried out on the IBM 360-91 at CCN, UCLA.

Swept Wing

The system considered in this example is an idealized swept wing structure (see Fig. 5). The structure is taken to be symmetric with respect to its middle surface. The upper half of the swept wing is modeled using 60 constant strain triangular (CST) elements to represent the skin and 70 symmetric shear panel (SSP) elements for the vertical webs. Extensive but plausible design variable linking is employed and the total number of independent design variables after linking is 18, 7 for the skin thicknesses (Fig. 5a) and 11 for the vertical webs (Fig. 5b). The wing is subjected to two distinct loading conditions and the material properties are representative of a typical aluminum alloy. Detailed input for this problem including material properties, initial design, nodal coordinates, applied nodal loadings, and constraint specification will be found in Tables 18-20 of Ref. 6.

The minimum weight optimum design of this idealized swept wing structure is sought subject to the following constraints: 1) tip deflection is not to exceed 152.4 cm (60 in.), 2) Von Mises equivalent stress is not to exceed 172,375 kPa

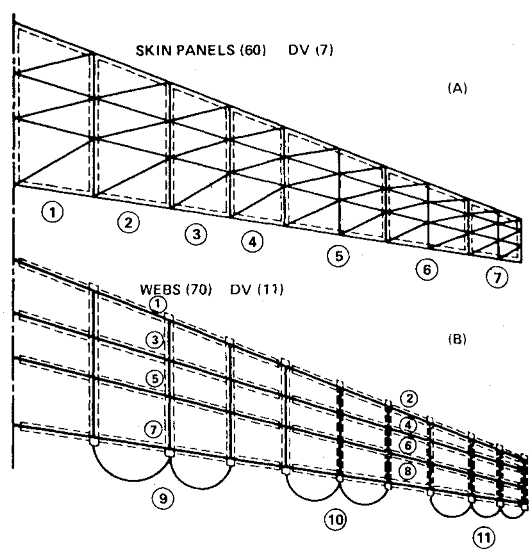

Fig. 5 Swept wing model.

(25,00 psi) in any finite element, 3) minimum gage of skin and web material is not to be less than 0.0508 cm (0.020 in.). This problem is treated as a pure discrete variable problem using the DUAL 1 algorithm. The available discrete values correspond to the standard gage sizes for 2024 aluminum alloy sheet metal, namely:

cm (in.)	cm (in.)
0.0508 (0.020)	0.2286 (0.090)
0.0635 (0.025)	0.2540 (0.100)
0.08128 (0.032)	0.3175 (0.125)
0.1020 (0.040)	0.4064 (0.160)
0.1270 (0.050)	0.4826 (0.190)
0.16002 (0.063)	0.6350 (0.250)
0.18034 (0.071)	0.79502 (0.313)
0.2032 (0.080)	0.9525 (0.375)

Table 1 Swept wing example—final designs

Linked design variable region	Thickness/minimum gage DUAL 2 (pure continuous)	DUAL 1 (pure discrete)	Rounding up
Skin			
1	10.21	12.5	12.5
2	8.89	9.5	9.5
3	7.85	8.0	8.0
4	6.47	8.0	8.0
5	5.795	6.25	6.25
6	5.16	4.5	6.25
7	1.00	1.0	1.0
Web			
1	1.343	1.25	1.6
2	1.107	1.25	1.25
3	2.807	3.15	3.15
4	1.785	1.6	2.0
5	9.870	9.5	12.5
6	1.523	1.25	1.6
7	4.668	5.0	5.0
8	3.935	3.55	4.0
9	1.503	1.25	1.6
10	2.524	2.0	3.15
11	3.335	2.15	3.55
Skin wt./tot. init. wt.	0.4400	0.4917	0.5018
Web wt./tot. init. wt.	0.0570	0.0552	0.0673
Tot. wt./tot. init. wt.	0.4970	0.5469	0.5692
No. of analyses	5	6	—
CPU time, s			
Total	19.3	25.6	—
Analysis	17.7	21.1	—
Optimizer	0.5	3.0	—

Note: Minimum gage = 0.0508 cm (0.020 in.), total initial weight = 2249 kg (4959 lb).

Table 2 Delta wing example—iteration history data

Analysis No.	DUAL 2 (continuous) W_n/W_I	$\omega_n/\bar{\omega}$	DUAL 1 (mixed case) W_n/W_I	$\omega_n/\bar{\omega}$
1	1.000	1.415	1.000	1.415
2	0.246[a]	1.008	0.240	1.005
3	0.193[a]	0.981	0.198[a]	1.000
4	0.165[a]	0.969	0.177[a]	0.987
5	0.171	1.004	0.173[a]	1.000
6	0.161[a]	0.994	0.166[a]	0.997
7	0.159	1.002	0.162[a]	0.998
8	0.157	1.004	0.160	1.002
9	0.154	1.003	0.156[a]	0.999
10	0.152	1.002	0.154[a]	1.002
11	0.151	1.001	0.154[a]	1.005
12	0.150[a]	1.001	0.153	1.002
13	0.149	1.001	0.153	1.000
14	0.148	1.000		
15	0.148	1.000		
CPU time, s				
Total	261		253	
Analysis	252		234	
Optimizer	2		13	

Note: W_I = total initial weight = 39,381 kg (86,820 lb), $\bar{\omega}$ = minimum required fundamental frequency = 2 Hz. [a] Infeasible design.

Table 3 Delta wing example—final designs

Linked design variable region	Fiber orientation, deg	Thickness/minimum gage Initial design	DUAL 2 (continuous)	DUAL 1 (mixed)
1	0	30	1.56	2
	±45	30	1.00[a]	2
	90	30	1.00[a]	2[a]
2	0	120	9.38	8
	±45	100	1.00[a]	2[a]
	90	20	1.00[a]	2[a]
3	0	300	18.16	20
	±45	200	2.56[a]	4[a]
	90	60	1.00[a]	2[a]
4	0	300	232.0	220
	±45	200	4.78[a]	6[a]
	90	60	1.60	2[a]
5	0	120	4.18	4
	±45	100	1.00	2
	90	20	1.00[a]	2[a]
6	0	300	11.12	14
	±45	200	2.18	2
	90	60	1.00[a]	2[a]
7	0	300	165.86	166
	±45	200	2.58	4
	90	60	2.36	2
8	0	40	1.00	2
	±45	40	1.00	2
	90	20	1.00	2
9	0	200	6.10	8
	±45	100	3.54	4
	90	40	1.00	2
10	0	200	125.12	122
	±45	100	5.02	6
	90	40	1.00	2
11	0	200	2.66	4
	±45	100	4.48	6
	90	40	1.00	2
12	0	200	79.78	78
	±45	100	7.72	8
	90	40	1.00	2
13	0	40	1.00	2
	±45	40	3.06	4
	90	20	1.00	2
14	0	60	42.64	42
	±45	20	9.32	10
	90	20	1.00	2
15	0	60	17.90	18
	±45	20	8.88	10
	90	20	1.00	2[a]
16	0	20	5.26	6
	±45	20	2.48[a]	2
	90	20	1.00[a]	2
Skin wt./tot. init. wt.			0.1334	0.1385
Web wt./tot. init. wt.			0.0142	0.0145
Tot. wt./tot. init. wt.			0.1476	0.1530
No. of analyses			15	13

Note: Minimum gage = 0.0127 cm (0.005 in.). Total initial weight = 39,381 kg (86,820 lb). [a] Transverse tension strain in bottom skin within 5% of limiting value.

With DUAL 1 only six analyses are needed to obtain the pure discrete variable solution for the swept wing example problem. The sequence of weights is as follows: 2249 (4959), 1235 (2722) [an infeasible design], 1245 (2744), 1238 (2730) 1230 (2712), and 1230 kg (2712 lb). Final design results are presented in Table 1. The skin and web thickness in Table 1 are given as multiples of the minimum gage thickness [0.0508 cm (0.020 in.)] and the final design weight information is expressed as a fraction of the initial weight which is 2249 kg (4959 lb). Pure continuous variable results, obtained with the DUAL 2 algorithm described in Ref. 5, are included in Table 1 for comparison purposes. Another discrete variable design, deduced by rounding up all continuous variable thicknesses to the next available gage size, is also given. It is observed that the DUAL 1 algorithm yields a final design that is 4% lighter than the intuitively derived discrete variable design.

Delta Wing

The last example treated here is a thin (3% thickness ratio) delta wing with graphite epoxy skins and titanium webs. This problem has been previously studied in Refs. 5 and 8 using a pure continuous variable approach. The results reported here are obtained using the mixed discrete-continuous design variable capability of the DUAL 1 algorithm. The variables describing the laminated fiber composite skin design are discrete (more precisely, integer variables representing the number of plies) and the titanium web variables are treated as continuous. The idealized structure is symmetric with respect to its middle surface. The skins are assumed to be made up of 0°, ±45°, 90° high-strength graphite epoxy laminates. Material oriented at 0° has fibers running spanwise, material oriented at 90° has fibers running chordwise. The laminates are assumed to be balanced and symmetric and each composite element is represented by merging a stack of four constant strain triangular orthotropic (CSTOR) elements. Therefore, the skin is represented by $4 \times 63 = 252$ CSTOR elements while the isotropic webs are modeled using 70 symmetric shear panels (SSP) elements. According to the linking scheme depicted in Fig. 6, it can be seen that the total number of independent design variables is equal to 60 made up as follows: 16 for 0° material, 16 for ±45° material, 16 for 90° material, and 12 for web material. The wing is subject to a single static load condition that is roughly equivalent to a uniformly distributed loading of 6.89 kPa (144 lb/ft^2) acting in combination with a specified temperature change condition [−128.9°C (−200°F) for the skins and −73.3°C (−100°F) for the webs]. Detailed input for this problem including material properties, nodal coordinates, applied nodal loadings, fuel mass distribution, and constraint specification will be found in Tables 23-25 of Ref. 6.

The minimum weight optimum design of this idealized delta wing is sought subject to the following constraints: 1) tip deflection is not to exceed 254 cm (100 in.), 2) specified maximum tension and compression strains in the longitudinal and transverse directions as well as a specified maximum shear strain are not to be exceeded, 3) the fundamental natural frequency is required to be equal to or greater than 2 Hz (with fix masses simulating fuel in the wing), 4) minimum gage sizes of 0.0127 cm (0.005 in.) for the fiber composite material and 0.0508 cm (0.020 in.) for the titanium web material. This problem is treated as a mixed discrete-continuous variable problem using the DUAL 1 algorithm. Iteration history data, expressed as a fraction (W_n/W_I) of the total initial weight [$W_I = 39{,}381$ kg (86,820 lb)], are presented in Table 2. In Table 3 initial and final design skin thicknesses are given as multiples of the minimum gage [0.0127 cm (0.005 in.)] which corresponds to the nominal thickness of a single ply of fiber composite material. Since the skin laminates are assumed to be balanced and symmetric, the smallest discrete change in lamina thickness is necessarily equal to two plies [0.0254 cm (0.010 in.)]; furthermore, the material thickness in the +45° direction is made equal to that in the −45° direction by design variable linking. It follows that the set of available discrete values for the thickness of the skin lamina is given by [0.0254 (0.010), 0.0508 (0.020), 0.0762 cm (0.030 in.), etc.]. Pure continuous variable results, obtained with the DUAL 2 algorithm described in Ref. 5, are included in Tables 2 and 3 for comparison purposes. Since the fundamental natural frequency constraint is the main design driver in this example, its iteration history expressed as a

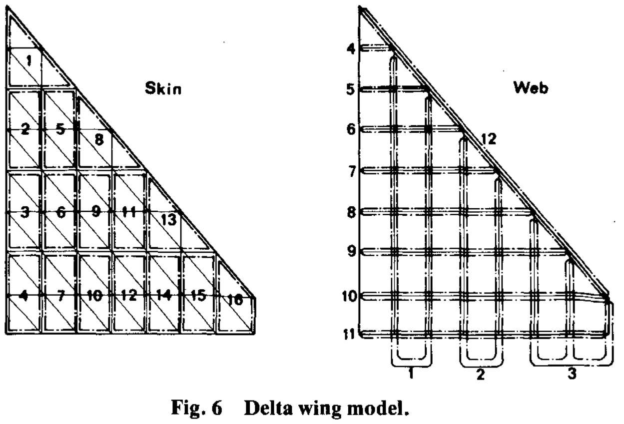

Fig. 6 Delta wing model.

fraction $(\omega_n/\bar{\omega})$ of the minimum required fundamental frequency [$\omega \geq \bar{\omega} = 2$ Hz] is given in Table 2, along with the normalized weight iteration history. It is interesting to note that DUAL 1 obtains a solution to the mixed-variable problem in fewer stages (and less time) than DUAL 2 requires to obtain the pure continuous variable solution.

Examination of the final design information given in Table 3 reveals that most of the fiber composite material is oriented in the spanwise direction. The design is governed primarily by the frequency constraint, however some strain constraints are critical in the skin of regions 1-6 and 15 (see Table 3). Since the web weight (see Table 3) represents only about 10% of the total structural weight, the web material distribution is omitted here; however, it can be found in Table 28 of Ref. 6.

Conclusions

It has been shown that the combined use of approximation concepts and dual methods can be successfully extended to provide a remarkably efficient minimum weight design optimization capability for sizing problems which involve discrete variables. This development is important because the description of fiber composite laminates and conventional sheet metal material (standard gage sizes) naturally involves discrete design variables. The DUAL 1 algorithm presented here has been devised to handle the general case of mixed discrete-continuous design variables and it treats either pure discrete or pure continuous variable problems as special cases. It should be noted however that for pure continuous variable problems the DUAL 2 algorithm, set forth in Ref. 5, is preferred since it is somewhat more efficient in the special case of pure continuous variables.

The basic reasons underlying the efficiency achieved by the DUAL 1 algorithm are:

1) The major computational effort in the optimizer is carried out in dual space on a continuous dual function, subject only to simple nonnegativity constraints, even when the approximate primal problem involves eiscrete variables.

2) The dimensionality of each dual space, namely, the number of critical and potentially critical constraints retained during that stage (Q_R), is relatively small for many problems of practical interest.

3) The operational dimensionality of the dual space tends to shrink rapidly after the first stage, because the set of nonzero λ_q's at the end of each stage is used to locate the starting point in the next stage.

4) Finally, by seeking the exact solution to each approximate primal problem, convergence of the overall design process is accelerated, although some intermediate designs in the iteration history may be infeasible.

The approach followed in this paper exploits the special algebraic structure of the approximate primal problem generated at each stage [see Eqs. (1-5)]. Since each approximate primal problem is separable and algebraically simple it is possible to construct an explicit and continuous dual function [see Eqs. (15, 7, and 14)]. This dual function exhibits discontinuous first derivatives on specific hyperplanes [see Eq. (12)] in the dual space, associated with discrete value shift in a primal variable, as well as discontinuous second derivatives on hyperplanes associated with a change in status (from free to bound) of a continuous primal variable [see Eqs. (19)].

The key to the success of the DUAL 1 algorithm resides in the method presented for coping with the first-order discontinuities of the dual function [see Eqs. (17 and 18)]. It has been found that projecting any one of the multiple gradients into the subspace defined by the current set of discontinuity hyperplanes yields a unique move direction Z along which the directional derivative $d\ell/dz$ is unique and positive [see Eqs. (31) and (34)]. Another important feature of the DUAL 1 algorithm is that it uses the intercepts of the move direction with first- and second-order discontinuity planes to accelerate solution of the one-dimensional maximizations (ODMs) and to facilitate recovery of the primal variables.

It should be recognized that when discrete variables are introduced, the approximate primal problem is no longer convex and, therefore, the dual formulation does not necessarily yield the true optimum design. Nevertheless, the computational experience reported here and in Ref. 6 confirms the observations made in Refs. 11 and 12 to the effect that although this extension of the dual formulation to discrete variables lacks mathematical rigor, it frequently gives useful and plausible results. The method presented in this paper represents a structural synthesis capability for mixed and pure discrete sizing problems which exhibits computational efficiency approaching those achievable in pure continuous variable problems.

Acknowledgments

This research was supported by NASA Research Grant NSG 1490.

References

[1]Cella, A. and Soosaar, K., "Discrete Variables in Structural Optimization," *Optimum Structural Design—Theory and Applications*, R. H. Gallagher and O. C. Zienkiewicz, eds., John Wiley and Sons, New York, 1973, Chap. 11.

[2]Toakley, A. R., "Optimum Design Using Available Sections," *Journal of Structures Div., ASCE*, Vol. 94, May 1968, pp. 1219-1241.

[3]Reinschmidt, K. F., "Discrete Structural Optimization," *Journal of Structures Div., ASCE*, Vol. 97, Jan. 1971, pp. 133-156.

[4]Cella, A. and Logcher, R. D., "Automated Optimum Design from Discrete Components," *Journal of Structures Div., ASCE*, Vol. 97, Jan. 1971, pp. 175-190.

[5]Schmit, L. A. and Fleury, C., "Structural Synthesis by Combining Approximation Concepts and Dual Methods," *AIAA Journal*, Vol. 18, Oct. 1980, pp. 1252-1260.

[6]Fleury, C. and Schmit, L. A., "Dual Methods and Approximation Concepts in Structural Synthesis," NASA CR-3226, 1980.

[7]Fox, R. L., *Optimization Methods for Engineering Design*, Addison-Wesley, Reading, Mass., 1971.

[8]Schmit, L. A. and Miura, H., "An Advanced Structural Analysis/Synthesis Capability—ACCESS 2," *International Journal of Numerical Methods in Engineering*, Vol. 12, Feb. 1978, pp. 353-377.

[9]Miura, H. and Schmit, L. A., "ACCESS 2 Approximation Concepts Code for Efficient Structural Synthesis—User's Guide," NASA CR-158949, Sept. 1978.

[10]Fleury, C. and Schmit, L. A., "ACCESS 3 Approximation Concepts Code for Efficient Structural Synthesis—User's Guide," NASA CR-159260, 1980.

[11]Ladson, L. S., *Optimization Theory for Large Systems*, MacMillan, New York, 1970, Chap. 8.

[12]Fisher, M. L., Northup, W. D., and Shapiro, J. F., "Using Duality to Solve Discrete Optimization Problems: Theory and Computational Experience," *Mathematical Programming Study*, Vol. 3, North-Holland Publishing Co., New York, 1975, pp. 56-94.